Free-energy functional for the Sherrington-Kirkpatrick model: The Parisi formula completed

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The Parisi formula for the free energy of the Sherrington-Kirkpatrick model is completed to a closed-form generating functional. We first find an integral representation for a solution of the Parisi differential equation and represent the free energy as a functional of order parameters. Then, we set stationarity equations for local maxima of the free energy determining the order-parameter function on interval [0, 1]. Finally, we show without resorting to the replica trick that the solution of the stationarity equations leads to a marginally stable thermodynamic state.

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I. INTRODUCTION

Interest of physicists in spin glasses has not subsided during last few decades. In spite of a tremendous progress in understanding of, in particular, the mean-field theory of spin glasses achieved in recent years, answers to a number of questions about physical properties of spin-glass models have not yet been found with ultimate validity. Although presently the major theoretical effort concentrates on clarifying the relevance of the mean-field solution for finite-dimensional systems, there still remain unresolved issues on the mean-field level.

The paragon mean-field theory for spin glasses is provided by the Sherrington-Kirkpatrick (SK) model introduced more than thirty years ago.[1] It took only a few years before Parisi inferred the form of a consistent solution of this model by a specific ansatz.[2] Parisi's solution, however, did not mark the end but rather the beginning of interest of theorists in spin-glass systems. There were two major reasons for further investigations of properties of the SK model and the Parisi solution. First, although most of the properties of the Parisi solution indicated that it is exact for the SK model, no mathematical proof existed confirming this conclusion in that time. Second, the Parisi solution was derived via the formal replica trick and the order parameters necessary for its description were constructed from nonmeasurable mathematical objects. The proper physical meaning of the replica-symmetry breaking (RSB) in the spin-glass phase was initially unclear.

The principal breakthrough in the proof of the exactness of the Parisi solution was achieved only a few years ago by Guerra and Talagrand.[3,4] They succeeded in proving rigorously that the replica-symmetry scheme of Parisi represents simultaneously both a lower as well as an upper bound on the free energy of the SK model in the thermodynamic limit. The existence of the thermodynamic limit with the self-averaging property of the free energy had already been proved earlier.[5] Unfortunately, neither the Parisi formula for the free energy nor the construction of Guerra and Talagrand do provide equations determining uniquely the thermodynamic state in the spin-glass phase. Although Talagrand conjectures the existence of a unique macroscopic state,[6] Parisi expressed the free energy only in a loose form of a formal maximum in a large unspecified functional space of unknown order-parameter functions. The way how to practically approach the maximum and how does the order-parameter function look like remain presently unset. Moreover, not having an explicit representation for the free-energy stationary with respect to all order parameters, it is not apparent how to define physical quantities such as magnetic susceptibility, entropy, or specific heat.

An attempt to construct a generating functional for the Parisi differential equation was done in Ref. 7. A variational functional was proposed there by adding the Parisi differential equation and its initial condition to the Parisi free energy with the aid of functional Lagrange multipliers. Although equations for the order-parameter functions were derived in this way, no solution or integral representation was found. All derived equations and representations remained on the level of nonlinear differential equations of the Parisi-type with unspecified distribution functions. Moreover, the Lagrange multipliers, used as variational functions in the free energy of Ref. 7, are configurationally dependent. They depend on the instantaneous values of a random magnetic field replacing the random spin exchange of the original inhomogeneous model.

The aim of this paper is to give the Parisi formula for the free energy of the SK model explicit meaning by solving the Parisi differential equation and finding a closed-form free-energy functional of the order parameters without adding auxiliary new functions or parameters. Maximizing such a free-energy functional, unspecified in the Parisi formulation, then becomes a uniquely defined process of finding solutions to stationarity equations determining fully the actual form and values of all order parameters. The implicit definition of the Parisi free energy completed in this way results in an explicit functional containing the entire physical information. All physical quantities are derived from it by standard means (derivatives with respect to external sources) of statistical mechanics without referring to the replica trick and mathematical replicas. Moreover, our integral representation of the mean-field free energy opens an alternative way to systematic expansions and nonperturbative approximations to physical quantities in the low-temperature spin-glass phase.

II. FREE-ENERGY FUNCTIONAL FOR THE PARISI SOLUTION

Using the replica trick, Parisi expressed the free energy of the SK model as a functional of the order-parameter function

$q(x)$ for $x\in[0,1]$ generalizing the single Sherrington-Kirkpatrick order parameter $q = N^{-1}\Sigma_i m_i^2$, where m_i are local magnetizations. The free energy density in the Parisi solution is then expressed as[2]

$$F_T[q] = -\frac{\beta^2}{4}\left(1+\int_0^1 q(x)^2 dx - 2q(1)\right) - \int_{-\infty}^{\infty}\mathcal{D}\eta\, f[0,h+\eta\sqrt{q(0)}],$$

$$F_P = \max_{q(x)} TF_T[q], \tag{1}$$

where we used an abbreviation for a Gaussian differential $\mathcal{D}\eta \equiv d\eta e^{-\eta^2/2}/\sqrt{2\pi}$. The most difficult "interacting" part of the above free energy $f(x,h)$ is not known explicitly. It is merely characterized by a Parisi differential equation with an initial condition,

$$\frac{\partial f(x,h)}{\partial x} = -\frac{1}{2}\frac{dq}{dx}\left[\frac{\partial^2 f(x,h)}{\partial h^2} + x\left(\frac{\partial f(x,h)}{\partial h}\right)^2\right],$$

$$f(1,h) = \ln[2\cosh \beta h], \tag{2}$$

that the physical functional must obey. The physical solution for the free energy should then be constructed by picking up the function $1 \geqslant q(x) \geqslant 0$, being nondecreasing on interval [0, 1], so that functional $f(x,h)$ obeying Eq. (2) maximizes the free energy from Eq. (1). We, however, do not know whether the maximizing order-parameter function $q(x)$ obeys a specific equation, and if yes, how does the equation look like. The functional space on which we should search for $q(x)$ maximizing the free-energy functional [Eq. (1)] is also unspecified.

We can gain some insight into the phase space of the order-parameter functions from the Guerra–Talagrand construction. It relies on the so-called discrete replica-symmetry breaking scheme. The latter can be derived straightforwardly from demanding thermodynamic homogeneity of the resulting free energy. Thermodynamic homogeneity is tested by stability of free energies with multiply replicated spin systems with respect to a weak interaction between the replicated spins.[8] We replicate the original system so many times until we reach thermodynamic homogeneity, that is, independence of a further replication. In this way a hierarchical structure of free energy emerges due to successive replications of the original system. The averaged free energy density with K hierarchies can then be represented as a functional of local response functions to the inter-replica interaction,[8]

$$\begin{aligned} f^K(q,\Delta\chi_1,\ldots,\Delta\chi_K;m_1,\ldots,m_K) &= -\frac{1}{\beta}\ln 2 + \frac{\beta}{4}\sum_{l=1}^{K} m_l\Delta\chi_l\left[2\left(q+\sum_{i=l+1}^{K}\Delta\chi_i\right)+\Delta\chi_l\right] \\ &\quad - \frac{\beta}{4}\left(1-q-\sum_{l=1}^{K}\Delta\chi_l\right)^2 - \frac{1}{\beta}\int_{-\infty}^{\infty}\mathcal{D}\eta \ln Z_K. \end{aligned} \tag{3a}$$

We used a sequence of partition functions,

$$Z_l = \left[\int_{-\infty}^{\infty}\mathcal{D}\lambda_l Z_{l-1}^{m_l}\right]^{1/m_l}, \tag{3b}$$

the initial condition for which reads $Z_0 = \cosh[\beta(h+\eta\sqrt{q} + \Sigma_{l=1}^K \gamma_1\sqrt{\Delta\chi_l})]$. We again denoted the Gaussian differential $\mathcal{D}\lambda$ introduced in Eq. (1). The order parameter q is the only one directly connected with local magnetizations. The other ones, $1 > \Delta\chi_1 \geqslant \Delta\chi_2 \geqslant \cdots \Delta\chi_K \geqslant 0$ and $1 \geqslant m_1 \geqslant m_2 \geqslant \cdots m_k \geqslant 0$, were introduced, since the linear response theory for the inter-replica interaction breaks down. All the order parameters are determined from stationarity equations locally maximizing free energy [Eq. (3a) and (3b)]. The number of hierarchies K is not an order-parameter characterizing a saddle point of the free energy. It is fixed from stability conditions, that is, it is a number of steps needed for achieving thermodynamic homogeneity.[8]

It was actually the discrete form of the replica-symmetry breaking that was used by Guerra and Talagrand to prove its exactness for the free energy of the SK model. They proved that free energy [Eq. (3a) and (3b)] becomes exact for the set of pairs $\{m_1,\Delta\chi_1,m_2,\Delta\chi_2,\ldots,m_K,\Delta\chi_K\}$ for which it is maximal. Their approach, however, does not specify whether the set of the order parameters is finite or infinite, how the parameters should be distributed on the underlying interval [0, 1], or whether they obey specific (stationarity) equations. The extremum may well become a supremum reached only at the boundary of the multidimensional phase space.

It is important to realize that the discrete free energy [Eq. (3a) and (3b)] and the continuous one [Eqs. (1) and (2)] are not identical. First, the former has two sets of order parameters m_l and $\Delta\chi_l$ for $l=1,\ldots,K$, while the latter only one, $q(x)$ for $x\in[0,1]$. Second, the order parameters from the discrete hierarchical free energy are determined from stationarity equations unlike the Parisi free energy, where the equation for $q(x)$ is essentially unknown. Third, the discrete free energy generally does not obey the Parisi differential equation [Eq. (2)]. In fact, the Parisi free energy emerges from a specific limit of the discrete ones, namely when $K\to\infty$, $\Delta\chi_l = \Delta\chi/K \to dx$, and we neglect second and higher powers of $\Delta\chi_l$ with the fixed index l (see Ref. 8).

Parisi's solution is defined only on a subclass of measures considered by Guerra and Talagrand on which we look for a maximum (supremum). It seems that at least for the SK model, continuous measures of the Parisi solution form a complete space and the Parisi free energy determines the exact, marginally stable solution. We demonstrated it explicitly in the asymptotic region below the critical temperature of the spin-glass phase in zero magnetic field[9] and recently also in the nonzero magnetic field.[10] On the other hand, there are models, such as the Potts spin glass,[11] where a discrete one-step RSB appears to be stable on a finite temperature interval.[12]

We demonstrate in this paper that independently of where the absolute maximum of the RSB free energy may lie, we can always construct a solution with a continuous distribution of differences $\Delta\chi_l$ and a single order-parameter function $m(\lambda)$ on the defining interval [0, 1] determined from explicit equations for a local maximum of the free energy. To formulate the continuous free energy and to fix 0 and 1 as the end points of the underlying interval on which the order-parameter function is defined, we introduce two physical order parameters q and X. The former corresponds to $q(0)$ and the latter to $q(1)-q(0)$ in the Parisi solution. We do not use the sequence $1 \geqslant m_1, \geqslant \cdots m_K \geqslant 0$ to set the interval on which the order-parameter function is defined as in Eq. (1). We find it more convenient to reverse the choice and use $\Delta\chi_l = X d\lambda$ as the fundamental infinitesimal differential. Neglecting all higher than linear powers of $\Delta\chi_l$, unless accompanied by a compensating summation over the labeling indices, free energy [Eq. (3a) and (3b)] reduces to

$$f[q,X;m(\lambda)] = -\frac{\beta}{4}(1-q-X)^2 - \frac{1}{\beta}\ln 2 + \frac{\beta X}{2}\int_0^1 d\lambda m(\lambda)[q+X(1-\lambda)] - \frac{1}{\beta}\langle g(1,h + \eta\sqrt{q})\rangle_\eta, \tag{4}$$

where $\langle X(\eta)\rangle_\eta = \int_{-\infty}^{\infty}\mathcal{D}\eta X(\eta)$. The principal achievement of this paper is an explicit integral representation of the interacting free energy $g(1,h)$. Similarly to the derivation of the Parisi differential equation in Ref. 13, we drop all terms of order $O(d\lambda^2)$ in the discrete hierarchy of partition sums Z_l when performing the continuous limit and end up with an integral representation,

$$g(1,h) = \mathbb{E}_0(X,h;1,0)\circ g_0(h) \equiv \mathbb{T}_\lambda \exp\left\{\frac{X}{2}\int_0^1 d\lambda[\partial_{\bar{h}}^2 + m(\lambda)g'(\lambda;h+\bar{h})\partial_{\bar{h}}]\right\} \times g_0(h+\bar{h})\Big|_{\bar{h}=0}, \tag{5}$$

where we used prime to denote a derivative with respect to the magnetic field h, $g'(\lambda,h)\equiv\partial_h g(\lambda,h)$. To reach a closed form for the continuous free energy, we introduced a "time-ordering" operator $\mathbb{T}_\lambda$ ordering products of λ-dependent noncommuting operators from left to right in a λ-decreasing succession. The time-ordered exponential is then defined as

$$\mathbb{T}_\lambda \exp\left\{\int_0^1 d\lambda \hat{O}(\lambda)\right\} \equiv 1 + \sum_{n=1}^{\infty}\int_0^1 d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \cdots \times \int_0^{\lambda_{n-1}} d\lambda_n \hat{O}(\lambda_1)\cdots\hat{O}(\lambda_n).$$

Time-ordering operators are a standard tool in representing quantum many-body perturbation expansions. The initial condition for the λ evolution in Eq. (5) is the local free energy $g_0(h)=\ln[\cosh\beta h]$. Unlike the Parisi construction, we develop the solution on the defining interval from zero to one, that is, the initial condition is taken at $\lambda=0$.

It is straightforward to verify that the interacting part of the free energy $g(\lambda,h)$ obeys a Parisi-like equation,

$$\frac{\partial g(\lambda,h)}{\partial\lambda} = \frac{X}{2}\left[\frac{\partial^2 g(\lambda,h)}{\partial h^2} + m(\lambda)\left(\frac{\partial g(\lambda,h)}{\partial h}\right)^2\right]. \tag{6}$$

The opposite overall sign of the right-hand side of this equation compared with Eq. (2) is connected with the reverted evolution of the initial condition used here.

Evolution operator $\mathbb{E}_0(X;\mu,\nu)$ contains only polynomials in powers of derivatives with respect to an auxiliary magnetic field replacing the Gaussian integration over auxiliary random fields λ_l in Eq. (3a) and (3b). It is, however, a nonlinear operator and that is why we must find analogous integral representations for functions $g'(\lambda,h)$ and $g''(\lambda,h)$ appearing in Eq. (6).

From the definition of the evolution operator $\mathbb{E}_0$, we obtain directly

$$\frac{\partial g(\lambda,h)}{\partial h} = \mathbb{E}_0(X,h;\lambda,0)\circ g_0'(h) + \frac{X}{2}\int_0^\lambda d\nu m(\nu)\mathbb{E}_0(X,h;\lambda,\nu)\circ[g'(\nu,h)\partial_h g'(\nu,h)]. \tag{7}$$

A solution to this integral equation can again be represented via an evolution operator and the $\mathbb{T}$-ordered exponential,

$$g'(\nu,h) = \mathbb{E}(X,h;\nu,0)\circ g_0'(h) \equiv \mathbb{T}_\lambda \exp\left\{X\int_0^\nu d\lambda\left[\frac{1}{2}\partial_{\bar{h}}^2 + m(\lambda) \times g'(\lambda;h+\bar{h})\partial_{\bar{h}}\right]\right\} g_0'(h+\bar{h})\Big|_{\bar{h}=0}. \tag{8}$$

Analogously, we obtain a differential equation for the second derivative of the free energy,

$$\frac{\partial g'(\lambda,h)}{\partial h} = \mathbb{E}(X,h;\lambda,0)\circ g_0''(h) + X\int_0^\lambda d\nu m(\nu)\mathbb{E}(X,h;\lambda,\nu)\circ g''(\nu,h)^2. \tag{9}$$

Its explicit solution is

$$g''(\nu,h) = \mathbb{E}_2(X,h;\nu,0)\circ g_0''(h) \equiv \mathbb{T}_\lambda \exp\left\{X\int_0^\nu d\lambda\left[\frac{1}{2}\partial_{\bar{h}}^2 + m(\lambda) \times \partial_{\bar{h}} g'(\lambda;h+\bar{h})\right]\right\} g_0''(h+\bar{h})\Big|_{\bar{h}=0}. \tag{10}$$

III. STATIONARITY EQUATIONS AND STABILITY

Having an explicit representation for the free energy, we can derive stationarity equations for its local extrema. Free energy $f[q,X;m(\lambda)]$ is a function of static parameters q and X and a nonlinear functional of the dynamical order-parameter function $m(\lambda)$. Vanishing of the free energy with respect to infinitesimal variations of the static parameters leads to the following equations:

$$q=\frac{1}{\beta^2}\langle g'(1,h_\eta)^2\rangle_\eta, \tag{11a}$$

$$X=\frac{1}{\beta^2}[\langle\mathbb{E}(X,h_\eta;1,0)\circ g_0'(h_\eta)^2\rangle_\eta-\langle g'(1,h_\eta)^2\rangle_\eta]. \tag{11b}$$

We denoted $h_\eta\equiv h+\eta\sqrt{q}$. Vanishing of the free energy with respect to infinitesimal variations of function $m(\lambda)$ leads to a functional equation,

$$\lambda=\frac{1}{\beta^2X}[\langle\mathbb{E}(X,h_\eta;1,0)\circ g_0'(h_\eta)^2\rangle_\eta - \langle\mathbb{E}(X,h_\eta;1,\lambda)\circ g'(\lambda,h_\eta)^2\rangle_\eta], \tag{11c}$$

valid for any $\lambda\in[0,1]$. Notice that Eq. (11c) for $\lambda=0$ is trivial and for $\lambda=1$ coincides with Eq. (11b). Hence, only equations for $0<\lambda<1$ serve for the determination of m as a function of the evolution parameter λ.

Free energy [Eq. (4)] complemented with stationarity equations [Eq. (11a)–(11c)] defines a thermodynamic state of the SK model for all input parameters. It is thermodynamically consistent so that physical values of all internal parameters specifying the thermodynamic state are determined self-consistently from stationarity equations and the standard thermodynamic relations hold. For instance, magnetic susceptibility reads

$$\chi_T=\frac{1}{\beta}\langle g''(1,h+\eta\sqrt{q})\rangle_\eta. \tag{12}$$

We do not have an integral representation such as in the Parisi replica-trick formulation,[2] since we do not use $q(x)$ as the order-parameter function but rather $m(\lambda)$. Nevertheless, we have an alternative implicit representation of $g''(\lambda,h)$ in Eq. (10). It is, however, important that we do not need to resort to the replica trick to define and calculate physical quantities in the thermodynamic state described by free energy [Eq. (4)].

One of the attractive features of the presented extension of the Parisi free energy is a possibility to verify stability of the solution of Eq. (11a)–(11c). We found earlier in Refs. 8 and 14 that the discrete K RSB solution is (marginally) stable if

$$1\geqslant\beta^2\left\langle\left\langle\left\langle 1-t^2+\sum_{i=1}^{l}m_i(\langle t\rangle_{i-1}^2-\langle t\rangle_i^2)\right\rangle_l^2\right\rangle_K\right\rangle_\eta \tag{13}$$

for $l=0,1,\ldots,K$. Here, we denoted $\rho_l(\eta;\lambda_K,\ldots,\lambda_l)=Z_l^{m_l}/\langle Z_l^{m_l}\rangle_{\lambda_l}$, $t\equiv\tanh[\beta(h+\eta\sqrt{q}+\Sigma_{l=1}^K\lambda_l\sqrt{\Delta\chi_l})]$, and $\langle t\rangle_l(\eta;\lambda_K,\ldots,\lambda_{l+1})=\langle\rho_l\cdots\langle\rho_1 t\rangle_{\lambda_1}\cdots\rangle_{\lambda_l}$ with $\langle X(\lambda_l)\rangle_{\lambda_l}=\int_{-\infty}^{\infty}\mathcal{D}\lambda_l X(\lambda_l)$. In the continuous limit when neglecting terms of order $O(\Delta\chi_l^2)$, we then find[10]

$$\langle\rho_l\langle t\rangle_{l-1}^2\rangle_{\lambda_l}-\langle\rho_l\langle t\rangle_{l-1}\rangle_{\lambda_l}^2\to\Delta\chi_l\langle t\rangle_{l-1}'^2. \tag{14}$$

In our notation $\lim_{K,l\to\infty}\Delta\chi_l/K=Xd\lambda$ and inserting Eq. (14) into Eq. (13), we obtain a continuous version of the stability conditions,

$$1\geqslant\frac{1}{\beta^2}\left\langle\mathbb{E}(X,h_\eta;1,\lambda)\circ\left[\mathbb{E}(X,h_\eta;\lambda,0)\circ g_0''(h_\eta) + X\int_0^\lambda d\nu m(\nu)\mathbb{E}(X,h_\eta;\lambda,\nu)\circ g''(\nu,h_\eta)^2\right]^2\right\rangle_\eta = \frac{1}{\beta^2}\langle\mathbb{E}(X,h_\eta;1,\lambda)\circ g''(\lambda,h_\eta)^2\rangle_\eta, \tag{15}$$

holding for each $\lambda\in[0,1]$. We used Eq. (9) to derive the last equality on the right-hand side of Eq. (15). We thus derived a functional generalization of the de Almeida–Thouless stability condition.[15]

To see that the above set of conditions is marginally satisfied, we utilize the fact that Eq. (11c) holds for each $\lambda\in[0,1]$. Then, also a total derivative of both sides with respect to λ must equal everywhere on interval [0,1]. Employing properties of the evolution operator $\mathbb{E}$, we find

$$\frac{d}{d\lambda}\mathbb{E}(X,h;1,\lambda)\circ g'(\lambda,h)^2=-X\mathbb{E}(X,h;1,\lambda)\circ g''(\lambda,h)^2. \tag{16}$$

Using this result in Eq. (11c), we obtain

$$\beta^2=\langle\mathbb{E}(X,h_\eta;1,\lambda)\circ g''(\lambda,h_\eta)^2\rangle_\eta, \tag{17}$$

telling us that the stability conditions from Eq. (15) are just marginally satisfied for each $\lambda\in[0,1]$. We hence see that the Parisi solution of the SK model satisfying Eq. (3a) and (3b) is marginally stable with no negative eigenvalues of the nonlocal susceptibility or the spin-glass susceptibility everywhere in the low-temperature spin-glass phase.

IV. CONCLUSIONS

Free energy [Eq. (4)] with the interacting part $g(\lambda,h)$ from Eq. (5) expressed via the evolution operator $\mathbb{E}_0$ contains a derivative of the desired solution $g'(\lambda,h)$. This cannot be avoided, since the Parisi differential equation [Eq. (6)] is nonlinear. Due to the time-ordering products used in the evolution operator, we can, nevertheless, use free energy [Eq. (4)] to develop controlled approximate schemes of computation of the free energy and other physical quantities. The most straightforward way is to expand the $\mathbb{T}$-ordered exponential in powers of the exponent. This practically corresponds to a power-series expansion of the order-parameter function $m(\lambda)$. Such an approach becomes asymptotically exact near the de Almeida–Thouless instability line.[10] We hence can analyze the critical behavior without the necessity to come over to a truncated model. Further on, we can use higher orders of the power expansion of the order-parameter function $m(\lambda)$ to systematically improve the asymptotic solution and extend it from the critical region below the instability line to a rather accurate approximation in the entire spin-glass phase. Another method for resolving the evolution operator is to approximate the order-parameter function with piecewise constant functions. In this way, we approximate the continuous scheme by a discrete one, resembling the discrete RSB from Eq. (3a) and (3b). These approximate solutions of free-energy functional [Eq. (4)] hence offer an alternative to expansions[16] of the variational free-energy functional from Ref. 7 in the effort to understand physical properties of the full RSB solution.

To conclude, we completed the Parisi formula for the free energy of the SK model so that it acquires the standard form demanded by the fundamental principles of statistical mechanics. The derived free energy is a function of two numerical order parameters q and X and a functional of an order-parameter function $m(\lambda)$ defined on interval $[0,1]$. The physical values of these order parameters are determined from stationarity equations for local extrema (maxima) of the free energy. The free energy thus becomes a well-defined generating functional from which all physical quantities can be derived via standard thermodynamic methods. There is no need to resort to the replica trick and a representation via mathematical replicas to identify measurable quantities. The integral representation of the solution of the Parisi differential equation demonstrates that the thermodynamic state of the SK model is marginally stable and allows for explicit systematic and nonperturbative approximations of the thermodynamics of mean-field spin-glass models.

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[1] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1972 (1975).
[2] G. Parisi, J. Phys. A **13**, L115 (1980).
[3] F. Guerra, Commun. Math. Phys. **233**, 1 (2003).
[4] M. Talagrand, Ann. Math. **163**, 221 (2006).
[5] F. Guerra and F. L. Toninelli, Commun. Math. Phys. **230**, 71 (2002).
[6] M. Talagrand, Ann. Math. **163**, 221 (2006).
[7] H. J. Sommers and W. Dupont, J. Phys. C **17**, 5785 (1984).
[8] V. Janiš, Phys. Rev. B **71**, 214403 (2005).
[9] V. Janiš and A. Klíč, Phys. Rev. B **74**, 054410 (2006).
[10] V. Janiš, A. Klíč, and M. Ringel, arXiv:0711.3384, J. Phys. A (to be published).
[11] D. Elderfield and D. Sherrington, J. Phys. C **16**, L497 (1983).
[12] D. J. Gross, I. Kanter, and H. Sompolinsky, Phys. Rev. Lett. **55**, 304 (1985).
[13] B. Duplantier, J. Phys. A **14**, 283 (1981).
[14] V. Janiš, Phys. Rev. B **74**, 054207 (2006).
[15] J. R. L. de Almeida and D. J. Thouless, J. Phys. A **11**, 983 (1978).
[16] A. Crisanti and T. Rizzo, Phys. Rev. E **65**, 046137 (2002).